EXACT TREATMENT OF ANTENNA CURRENT WAVE REFLECTION

AT THE END OF A TUBE-SHAPED CYLINDRICAL ANTENNA

by

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Technical Report No. 5

CALIFORNIA INSTITUTE OF TECHNOLOGY

Pasadena, California

A report on research conducted under

contract with the Air Force Office of Scientific Research

March 1955

INTRODUCTION

Antennas, whose radiating body has a cross section which is small in comparison to the length of the antenna and to the wave length, i. e. antennas consisting of wires and rods, are to a very wide extent treated with the help of a linearized integral equation, the invention of the author (Hallén 1930, eqn. 20a, b; Hallén 1938, eqn. 24; Hallén 1953, eqn. 35.31). In this equation the distance between two points on the antenna is normally represented by the distance between the corresponding points on some central line. Only when the distance is small this is not permitted and from such regions arises the only term which contains the dimension of the cross section, which is a parameter mainly consisting of a logarithm. The equation therefore has a certain limited degree of accuracy which is such that the ratio of the radius of cross section to the length of the antenna or to the wave length is neglected compared with unity. The results which can be drawn from the linearized integral equation thus also should have this limited accuracy which is a normal one in electrotechnics in all kinds of devices, where wires are involved. Nevertheless much discussion has gone on about this accuracy. The only way of finding definite numerical answers to this question is to solve exactly the antenna integral equations, both the linearized one and the exact one, for some special case. Nowadays this can be done for a straight cylindrical tube-shaped antenna.

(R. Gans has recently (Gans 1953, Gans 1954) expressed the opinion that Hallén's linearized integral equation should not have any <u>exact</u> solution. This is a mistake made by Gans because he apparently has never seen my original papers. What he studies and criticizes is the coarser form of the equation given in many papers and books as "Hallén's integral equation".

Gans in reality criticizes the deviation that is made in those papers from my own form, which is not subject to any criticism of the kind expressed by Gans. See (Papas 1954) and (Borgnis und Papas 1955). It must also be remembered that even if I, to some extent, would agree with Gans if he had directed his criticism to the proper persons, it is for practical purposes not very essential if the coarser form of the equation is used in the beginning. Those authors have at a later stage in their deductions usually completed the linearization so that they in reality use the correct form even if it has not been written explicitly. As they try to find only an approximate solution it is even for this reason unessential if their equation is not quite the correct linearized one.)

Any electric field in free space, periodic with respect to time as $e^{j\omega t}$, can be expressed with the help of a vector potential $\underline{A}$ as follows:

$$\underline{E} = \frac{c}{j\beta} (\text{grad div } \underline{A} + \beta^2 \underline{A}) \qquad (1)$$

where c is the electromagnetic wave velocity and

$$\beta = \frac{\omega}{c}$$

a wave constant. On a cylindrical antenna oscillating under conditions which involve axial symmetry, the current and consequently its vector potential has the direction of the cylinder axis. If the ohmic resistance is neglected the tangential component of $\underline{E}$ vanishes in the surface of the antenna and from (1) follows the nowadays well known fact (Hallén 1930, p. 11; Hallén 1938, p. 14; Hallén 1953, p. 404) that the vector potential, and hence the scalar potential, of the antenna field is exactly sine-shaped along the surface of a cylindrical antenna. This gives the integral equation for the antenna current $I(x)$ of a cylindrical antenna, fed by a potential jump $2\,V_o$ in the middle (Hallén 1953, eqn. 35.15):

$$\int_{-\ell}^{\ell} d\xi \;\; I(\xi)\,\frac{1}{2\pi}\int_{0}^{2\pi} \frac{e^{-j\beta r}}{r}\,d\varphi = \frac{4\pi}{Z_o}\;(-V_o\,e^{-j\beta|x|} + A\cos\beta\,x) \qquad (3)$$

where 2ℓ is the length, a the cylinder-radius of the antenna, $r = \left[(x-\xi)^2 + 4\,a^2\sin^2\frac{1}{2}\varphi\right]^{\frac{1}{2}}$ the distance between two surface points, Z_o the wave resistance of free space ($Z_o = 377$ ohms) and A an unknown constant. The minus sign on V_o, which we have introduced here, although it was not in the previous papers, only indicates that we have turned the potential jump so that the outgoing potential wave is positive when travelling in the direction of negative x. In order that (3) shall be mathematically exact the ohmic resistance should be negligible and the cylinder should have no end-surfaces, i. e. it must be a thin tube. (In practice only the pieces nearest to the cylinder ends need to be hollow).

The current $I(x)$ can be considered either as a standing wave or as a system of travelling waves, one outgoing current wave travelling in both directions from the feeding point, and a series of current waves reflected at the ends of the antenna. The terms of the right side of eqn. (3) are, besides a constant factor, the corresponding terms of the vector potential along the antenna. Thus $-\frac{V_o}{c}\,e^{-j\beta|x|}$ is the outgoing vector potential wave and $\frac{A}{c}\cos\beta x$ the sum of all the reflected vector potential waves. The outgoing scalar potential wave is $+\,V_o\,e^{j\beta x}$ in the direction of negative x and $-\,V_o\,e^{-j\beta x}$ in the positive direction. The sum of all the reflected scalar potential waves is $-jA \sin \beta x$. Thus the travelling potential waves have constant amplitudes (on the surface of the cylindrical antenna), whereas the corresponding travelling current waves always decrease.

The outgoing travelling wave is easily found from eqn. (3). It is the same as the current on an infinite antenna without end reflections, hence $A = 0$ and $\ell = \infty$. Thus the outgoing current wave satisfies the integral equation (Hallén 1948a, eqn. 7; Hallén 1953, eqn. 35.17):

$$\int_{-\infty}^{\infty} d\xi \quad I(\xi)\, \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{-j\beta r}}{r}\, d\varphi = -\frac{4\pi}{Z_0} V_0\, e^{-j\beta|x|} \tag{4}$$

(For the solution of this equation see eqn. (9)). Before we proceed we will even write down the linearized equations, which correspond to (3) and (4). These are for an antenna of finite length (Hallén 1938, eqn. 24; Hallén 1953, eqn. 35.31):

$$I(x) \log \frac{4(\ell^2 - x^2)}{a^2} + \int_{-\ell}^{\ell} \frac{I(\xi)\, e^{-j\beta|x-\xi|} - I(x)}{|x-\xi|}\, d\xi = \frac{4\pi}{Z_0}\left(-V_0\, e^{-j\beta|x|} + A \cos \beta x\right) \tag{5a}$$

or in a modified form:

$$I(x)\left[\log \frac{4}{a^2\beta^2} - 2\gamma - j\pi - e^{-j\beta(\ell-x)}\, \mathcal{L}\,(\beta(\ell - x)) - e^{-j\beta(\ell+x)}\, \mathcal{L}\,(\beta(\ell + x))\right] +$$

$$+ \int_{-\ell}^{\ell} \frac{I(\xi) - I(x)}{|x-\xi|}\, e^{-j\beta|x-\xi|} d\xi = \frac{4\pi}{Z_0}\left(-V_0\, e^{-j\beta|x|} + A \cos \beta x\right) \tag{5b}$$

where the function $\mathcal{E}$ is the complex amplitude function of the sine and cosine integral as defined (and tabulated) in (Hallén 1948a; Hallén 1948b, eqn. 16; Hallén 1955, eqn. 9, table IV). From (5b) we immediately get the linearized integral equation for the outgoing waves, if we put $\ell = \infty$ and $A = 0$ (Hallén 1948a, eqn. 14; Hallén 1953, eqn. 35.35).

$$I(x)\left[\log\frac{4}{a^2\beta^2} - 2\gamma - j\pi\right] + \int_{-\infty}^{\infty}\frac{I(\xi) - I(x)}{|x-\xi|}\,e^{-j\beta|x-\xi|}d\xi = -\frac{4\pi}{Z_o}V_o\,e^{-j\beta|x|} \tag{6}$$

The difference between (6) and (4) is that in (6) the radius of the cross section of the antenna is consequently suppressed except in the first logarithmical term. There is the same connection between (5a, b) and (3). The linearized equations are much easier to handle, but that is not the main reason to use them as we can now even solve (3) and (6) exactly. The linearizing has been invented in order to make it possible to solve antenna problems of much more complicated nature. Even if the antenna has a curved instead of a straight central line, has variable cross section and a cross section which is not circular, integral equations can still be set up with only one unknown variable, of a type corresponding to (5a). The fact that in the general case the vector potential along the antenna is not sine-shaped, makes no obstacle. If even the ohmic resistance of the antenna is taken into account as well as an incoming outer field, this general linearized antenna equation can be written (Hallén 1930, eqn. 20a, b; Hallén 1938, eqn. 10, 13):

$$I(x) = \frac{1}{\Omega}\left\{ - I(x)\log\frac{\ell^2 - x^2}{\ell^2} - \int_{-\ell}^{\ell}\left(\frac{I(\xi)\,e^{-j\beta r(\xi,x)}}{r(\xi,x)} - \frac{I(x)}{|x-\xi|}\right)d\xi \;+\right.$$

$$+ \int_{-\ell}^{\ell} I(\xi)\, d\xi \int^{x} \left[\beta \sin \beta(x - s) \frac{1 - \cos(\xi, s)}{r(\xi, s)} e^{-j\beta r(\xi, s)} + \right.$$

$$\left. + \cos \beta(x - s) \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial s}\right) \frac{e^{-j\beta r(\xi, s)}}{r(\xi, s)}\right] ds -$$

$$- 2 \int^{x} I(s) \cos \beta(x - s) \frac{a'(s)}{a(s)}\, ds + j \frac{4\pi}{Z_o} \int^{x} I(s) \sin \beta(x - s) z_\beta(s)\, ds -$$

$$\left. - j \frac{4\pi}{Z_o} \int^{x} E_o(s) \sin \beta(x - s)\, ds + j \frac{4\pi}{Z_o} V_o \sin \beta |x - \ell_1| \right\} +$$

$$+ A \cos \beta x + \beta \sin \beta x \tag{5c}$$

where $\Omega = 2 \log \frac{2\ell}{a(x)}$ is a parameter.

Here x and ξ are the lengths of the arc of the central line of the antenna reaching from $-\ell$ to ℓ; $r(\xi, x)$ is the straight distance between two points on the central line (thus $r(x,x) = 0$) ; $a(x)$ is the radius of cross section at the point x , if circular, and the "equivalent radius" if not; $z\beta$ is the inner impedance per unit length of the antenna wire. $E_o(x)$ is the component along the antenna central line of an incoming outer electric field, if any, and $2\,V_o$ the driving potential jump (directed as in eqn. (3)), if any, applied in the point $x = \ell_1$; and A and B constants. The integrals with respect to s , which have upper limits x , have arbitrary, but constant lower limits. All integrands remain finite even when $\xi = x$ or $\xi = s$.

The equations (4) and (6) for the outgoing current waves both can easily be exactly solved. The general solution of both includes an undetermined solution to the corresponding homogeneous equations, i. e. the equations with the right side put equal to zero. In the case of the exact equation (4) this eigen-solution (see further below) is the wave guide solution for the tube:

$$I = \text{constant} \cdot e^{\pm \frac{x}{a}\sqrt{\xi_m^2 - a^2\beta^2}} \tag{7}$$

(Hallén 1953, p. 410), where the ξ_m are the zeros of the Bessel function of order 0. This solution in all ordinary antenna cases $(\lambda \gg a)$, when the wave length is big enough to make $\beta < \frac{\xi_m}{a}$, is aperiodic in space: frequency below cutoff frequency of the tube. In the case of the linearized antenna integral equation (6) we have (see further below) a corresponding eigen-solution to the homogeneous equation:

$$I = \text{constant} \cdot e^{\pm j \frac{x}{a}\sqrt{\frac{4}{\gamma_1^2} + a^2\beta^2}} \tag{8}$$

where $\gamma_1 = 1.781072$ or $\log \gamma_1 = \gamma = 0.577216$ is Euler's constant. Eqn. (8) represents travelling waves in both directions, which in all ordinary antenna cases $(\lambda \gg a)$ would have a very low velocity. These waves have no physical significance.

Thus the general solutions of both (4) and (6) are undetermined by terms of the kind indicated by (7) and (8). However, if we add the condition that the solutions to (4) and (6) should have symmetry with respect to the feeding point, so that the two waves, going out in both directions should be equivalent then the solution of (4) as well as of (6) is unique. This solution of the exact equation (4) is (Hallén 1948a, eqn. 8; Hallén 1953, eqn. 35.24):

$$I(x) = -\frac{4\pi}{Z_0} V_0 \frac{j\beta}{2\pi} \int_{\Gamma} \frac{e^{j\alpha x}}{(\alpha^2 - \beta^2)\, I_0\left(a\sqrt{\alpha^2 - \beta^2}\right) K_0\left(a\sqrt{\alpha^2 - \beta^2}\right)}\, d\alpha \tag{9}$$

and of the linearized equation (6):

$$I(x) = -\frac{4\pi}{Z_o} V_o \frac{j2\beta}{2\pi} \int_{\Gamma} \frac{e^{j\alpha x}}{(\alpha^2 - \beta^2) \log \frac{4}{a^2 \gamma_1^2 (\alpha^2 - \beta^2)}} d\alpha \tag{10}$$

where I_o and K_o (Watson 1944, p. 78 - 79) are the Bessel and Hankel functions of imaginary argument and Γ and integration path in the complex α-plane which follows the real axis from $-\infty$ to $+\infty$ but avoids $\alpha = \pm\beta$ as if β had a small negative imaginary part. If x is changed into $-x$, (9) and (10) remain unaltered.

Both (9) and (10) are exact solutions of respective (4) and (6), but they still do not represent exactly pure antenna waves. Both contain small terms of a type similar to (7) respective (8) but with the difference that x is substituted by $|x|$, i. e. (9) contains outgoing wave guide waves (below cutoff frequency) and (10) an extra outgoing very slowly travelling wave. Those additional parts in (9) and (10) come from the poles of the integrand. By changing the path of integration we can separate those small terms from the rest, which is the true antenna outgoing travelling wave, for (9), according to (Hallén 1953, eqn. 35.27 and 35.28a):

$$I(x) = -\frac{4\pi}{Z_o} V_o \psi(\beta x)\, e^{-j\beta|x|} \tag{11}$$

where

$$\psi(\beta x) = e^{j\beta|x|} \frac{2}{\pi^2} \left\{ \int_0^1 \frac{e^{-j\beta|x|\sqrt{1-u^2}}}{u\sqrt{1-u^2}\, H_o^{(1)}(a\beta u)\, H_o^{(2)}(a\beta u)} du + \right.$$

$$\left. + j \int_1^{\infty} \frac{e^{-\beta|x|\sqrt{u^2-1}}}{u\sqrt{u^2-1}\, H_o^{(1)}(a\beta u)\, H_o^{(2)}(a\beta u)} du \right\} \tag{12}$$

and in the case (10) (Hallén 1953, eqn. 35.36):

$$\psi(\beta x) = e^{j\beta|x|}\frac{2}{\pi^2}\left\{\int_0^1 \frac{e^{-j\beta|x|\sqrt{1-u^2}}}{u\sqrt{1-u^2}\left[1+\frac{4}{\pi^2}\log^2(\frac{1}{2}a\beta\gamma_1 u)\right]}du + \right.$$

$$\left. + j\int_1^\infty \frac{e^{-\beta|x|\sqrt{u^2-1}}}{u\sqrt{u^2-1}\left[1+\frac{4}{\pi^2}\log^2(\frac{1}{2}a\beta\gamma_1 u)\right]}du\right\} \tag{13}$$

(I repeat that the minus signs before V_o in eqn. (3), (4), (5), (6), (9), (10), (11) only indicate that whereas the current is counted positive in the direction of positive x, the potential jump $2\ V_o$ is directed so that it would drive current in the negative direction, i. e. the scalar potential wave is positive for negative x and vice versa.) Another way of separating the true antenna current from the total current follows from eqn. 17c, 17d. In both cases the function $\psi(\beta x)$ in spite of the first factor is perfectly aperiodic (diagrams in (Hallén 1953, p. 415)). In the case of the exact solution the expressions (9) and (11, 12) have each a distinct physical meaning: the expression (11, 12) is the true antenna current on the outside of the antenna tube, whereas (9) is the sum of this current and the wave guide current inside the tube. In the linearized case there is no such distinction. The new expressions (11, 12) and (13) still are exact solutions of the integral equations (4) and (6) but only on one side of the feeding point, either the side of positive x or of negative x_1 for the other side it is necessary to add an eigen-solution of the type (7) respective (8) to make the integral equation satisfied for all values of x. The expression (9) and (10) are the only symmetrical solutions to (4) and (6). We remark that the terms which make the difference between (9) and (11, 12) and between (10) and (11, 13) contain the small factor $a\beta$.

The close connection between (12) and (13) or between (9) and (10) is obvious. We have in the denominator of the integrands in (12)

$$H_o^{(1)}(z)\, H_o^{(2)}(z) = J_o^{2}(z) + Y_o^{2}(z) =$$

$$= 1 + \frac{4}{\pi^2} \log^2 \tfrac{1}{2} \gamma_1 z + z^2 \left\{ - \tfrac{1}{2} + \frac{2}{\pi^2} \left(\log \tfrac{1}{2} \gamma_1 z - \log^2 \tfrac{1}{2} \gamma_1 z \right) \right\} + \ldots$$

and thus we can get (13) from (12) by suppressing the powers of $a\beta$ but keeping it in the logarithm. This is just what we did when we put up the linearized integral equation (5a). The numerical agreement between the solution (12) of the exact equation (4) and the solution (13) of the linearized equation (6) is very high even near the feeding point, in all ordinary cases. Only for extremely thick antennas is there a difference. It now also appears that the consequent complete linearization of the integral equation (eqn. 5a) with suppressing of all powers of a but keeping a in the logarithm, which I undertook in (Hallén 1930 and Hallén 1938) is the only strict procedure. My followers have mostly written a fictitious $r = \sqrt{(x - \xi)^2 + a^2}$, the distance between a point on the surface and a point on the axis, instead of $|x - \xi|$, and even used a square root expression in the first term of (5a). In doing so they have not at all increased the accuracy and in fact have made (5a) unsolvable, in a strict mathematical sense, because the derivative of the left side then will remain continuous at $x = 0$, but not of the right side. Gans, in his above mentioned papers, criticizes this inconsistency, which he quite erroneously attributes to me. The remedy which he recommends, the changing of the right side of (5a) as well, does not seem adequate. It is better to return to the original for this integral equation method (Hallén 1938).

(When there is no potential jump, passive antennas, and the antenna is solid, the distance $r = \sqrt{(x-\xi)^2 + a^2}$ may be used, because one can use the vanishing electric field along the antenna axis to get an integral equation, but then there are always end surfaces.)

From the exact symmetrical expression (9) for the outgoing travelling current wave it is very easy to get the potentials and field strengths in any point outside the antenna and even inside the antenna tube. We only have to apply the formula:

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} \frac{e^{-j\beta r + j\alpha\xi}}{r}\, d\xi = \begin{cases} 2e^{j\alpha x}\, I_o(a\sqrt{\alpha^2-\beta^2})\, K_o(\rho\sqrt{\alpha^2-\beta^2}), & \rho > a \\ 2e^{j\alpha x}\, I_o(\rho\sqrt{\alpha^2-\beta^2})\, K_o(a\sqrt{\alpha^2-\beta^2}), & \rho < a \end{cases} \tag{14}$$

where $r = \sqrt{(x-\xi)^2 + \rho^2 + a^2 - 2\, a\, \rho \cos\varphi}$ is the distance between two points, whose cylinder coordinates are ρ, 0, x respective a, φ, ξ.
By putting (9) in the integral expression for the vector potential similar to the left side of (3) we get with the help of (14) the vector potential in any point:

$$A_x = -\frac{V_o}{c}\cdot\frac{j\beta}{\pi}\int_\Gamma \frac{e^{j\alpha x}}{\alpha^2-\beta^2}\cdot\frac{K_o(\rho\sqrt{\alpha^2-\beta^2})}{K_o(a\sqrt{\alpha^2-\beta^2})}\, d\alpha \quad , \quad \rho > a \tag{15a}$$

$$A_x = -\frac{V_o}{c}\cdot\frac{j\beta}{\pi}\int_\Gamma \frac{e^{j\alpha x}}{\alpha^2-\beta^2}\cdot\frac{I_o(\rho\sqrt{\alpha^2-\beta^2})}{I_o(a\sqrt{\alpha^2-\beta^2})}\, d\alpha \quad , \quad \rho < a \tag{15b}$$

$$A_x = -\frac{V_o}{c}\, e^{-j\beta|x|} \quad , \quad \rho = a \tag{15c}$$

As the scalar potential $V = \frac{jc}{\beta}\frac{\partial A_x}{\partial x}$ we get from (15):

$$V = V_o\,\frac{j}{\pi}\int_\Gamma \frac{\alpha\, e^{j\alpha x}}{\alpha^2-\beta^2}\cdot\frac{K_o(\rho\sqrt{\alpha^2-\beta^2})}{K_o(a\sqrt{\alpha^2-\beta^2})}\, d\alpha \quad , \quad \rho > a \tag{16a}$$

$$V = V_o \frac{j}{\pi} \int_{\Gamma} \frac{\alpha\, e^{j\alpha x}}{\alpha^2 - \beta^2} \cdot \frac{I_o\,(\rho\sqrt{\alpha^2 - \beta^2})}{I_o\,(a\sqrt{\alpha^2 - \beta^2})}\, d\alpha \quad , \quad \rho < a \tag{16b}$$

$$V = - V_o \frac{x}{|x|}\, e^{-j\beta|x|} \quad , \quad \rho = a \tag{16c}$$

As the magnetic field is $B\varphi = - \frac{\partial A_x}{\partial \rho}$ we also get from (15):

$$B\varphi = - \frac{V_o}{c} \cdot \frac{j\beta}{\pi} \int_{\Gamma} \frac{e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{K_1\,(\rho\sqrt{\alpha^2 - \beta^2})}{K_o\,(a\sqrt{\alpha^2 - \beta^2})}\, d\alpha \quad , \quad \rho > a \tag{17a}$$

$$B\varphi = \frac{V_o}{c} \cdot \frac{j\beta}{\pi} \int_{\Gamma} \frac{e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{I_1\,(\rho\sqrt{\alpha^2 - \beta^2})}{I_o\,(a\sqrt{\alpha^2 - \beta^2})}\, d\alpha \quad , \quad \rho < a \tag{17b}$$

$$B\varphi = \frac{V_o}{c} \cdot \frac{j\beta}{\pi} \int_{\Gamma} \frac{e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{K_1\,(a\sqrt{\alpha^2 - \beta^2})}{K_o\,(a\sqrt{\alpha^2 - \beta^2})}\, d\alpha \quad , \quad \rho = a + 0 \tag{17c}$$

$$B\varphi = \frac{V_o}{c} \cdot \frac{j\beta}{\pi} \int_{\Gamma} \frac{e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{I_1\,(a\sqrt{\alpha^2 - \beta^2})}{I_o\,(a\sqrt{\alpha^2 - \beta^2})}\, d\alpha \quad , \quad \rho = a - 0 \tag{17d}$$

The axial electric field component is $E_x = - \frac{jc}{\beta} \left(\frac{\partial^2 A_x}{\partial x^2} + \beta^2 A_x \right)$:

$$E_x = \frac{V_o}{\pi} \int_{\Gamma} e^{j\alpha x}\, \frac{K_o\,(\rho\sqrt{\alpha^2 - \beta^2})}{K_o\,(a\sqrt{\alpha^2 - \beta^2})}\, d\alpha \quad , \quad \rho > a \tag{18a}$$

$$E_x = \frac{V_o}{\pi} \int_{\Gamma} e^{j\alpha x}\, \frac{I_o\,(\rho\sqrt{\alpha^2 - \beta^2})}{I_o\,(a\sqrt{\alpha^2 - \beta^2})}\, d\alpha \quad , \quad \rho < a \tag{18b}$$

$$E_x = 0 \quad , \quad \rho = a \tag{18c}$$

The radial electric field component is $E_\rho = - \frac{\partial V}{\partial \rho}$:

$$E_\rho = V_o \frac{j}{\pi} \int_{\Gamma} \frac{\alpha\, e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{K_1\,(\rho\sqrt{\alpha^2 - \beta^2})}{K_o\,(a\sqrt{\alpha^2 - \beta^2})}\, d\alpha \quad , \quad \rho > a \tag{19a}$$

$$E_\rho = - V_o \frac{j}{\pi} \int_\Gamma \frac{\alpha\, e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{I_1 (\rho \sqrt{\alpha^2 - \beta^2})}{I_o (a \sqrt{\alpha^2 - \beta^2})} \, d\alpha \quad , \quad \rho < a \tag{19b}$$

$$E_\rho = V_o \frac{j}{\pi} \int_\Gamma \frac{\alpha\, e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{K_1 (a \sqrt{\alpha^2 - \beta^2})}{K_o (a \sqrt{\alpha^2 - \beta^2})} \, d\alpha \quad , \quad \rho = a + 0 \tag{19c}$$

$$E_\rho = - V_o \frac{j}{\pi} \int_\Gamma \frac{\alpha\, e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{I_1 (a \sqrt{\alpha^2 - \beta^2})}{I_o (a \sqrt{\alpha^2 - \beta^2})} \, d\alpha \quad , \quad \rho = a - 0 \tag{19d}$$

From (17c) we get the outside current, i. e. the true antenna current $I_e = B\varphi \, \frac{2\pi a}{\mu_o}$:

$$I_e = - \frac{4\pi}{Z_o} V_o \frac{ja\beta}{2\pi} \int_\Gamma \frac{e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{K_1 (a \sqrt{\alpha^2 - \beta^2})}{K_o (a \sqrt{\alpha^2 - \beta^2})} \, d\alpha \tag{20}$$

and from (17d) the inner current (wave guide current $I_i = - B_\varphi \, \frac{2\pi a}{\mu_o}$:

$$I_i = - \frac{4\pi}{Z_o} V_o \frac{ja\beta}{2\pi} \int_\Gamma \frac{e^{j\alpha x}}{\sqrt{\alpha^2 - \beta^2}} \cdot \frac{I_1 (a \sqrt{\alpha^2 - \beta^2})}{I_o (a \sqrt{\alpha^2 - \beta^2})} \, d\alpha \tag{21}$$

The expression (20) is identical with (11, 12) and the sum of (20) and (21) is identical with (9). In (17d, 19d, 21d) the integral can easily be carried out and gives a series of wave guide waves below cutoff frequency (cf. Hallén 1953, eqn. 35.25). The character of the potential jump 2 V_o in the feeding point and the direction we have given it in this investigation is clearly seen in eqn. (16c). In the following we will also give some other forms and asymptotic values for some of the quantities (15 - 21).

INTEGRAL EQUATION FOR REFLECTED ANTENNA CURRENT WAVES

After this survey of the outgoing current wave we turn to the main object, the reflected waves. The first investigation I made on them (Hallén 1948b) was based on the principle that a standing wave expression was first obtained and then this standing current wave was dissolved into a system of travelling waves. Thus the expression (Hallén 1948b, eqn. 55) for the first, second and following travelling waves was obtained. However, there has as yet only existed approximate expressions for the standing waves, consisting of a series of a limited number of terms, obtained by iteration from the integral equation (5a). Thus the travelling waves will also be known through series with a limited number of terms. This method is laborious, as already the third term in the standing current wave expression (Hallén 1948b, eqn. 30) is extremely complicated. A new direct method of finding the travelling waves has been developed later (Hallén 1953). It consists of splitting up the equation (5a) into a series of integral equations, one for each travelling wave. This can be done by studying the periodicity of both sides of (5a) with respect to x as well as to ℓ. Thus the series of integral equations (Hallén 1953, eqn. 35.61; 35.62a, b, c) for the outgoing wave, for the first, second, etc. reflected waves, is obtained. From these equations it is very easy to find the result eqn. (55) in the former paper, and even increase the number of terms, i. e. the accuracy, in the series solutions (Hallén 1953, eqn. 35.65, 35.66).

These new integral equations also invite us to try to solve the antenna problem exactly in the form of integrals without any abbreviated series. In fact, since Levine and Schwinger treated sound waves in a limited unflanged pipe (Levine and Schwinger 1948) with the help of Hopf's and Wiener's solution to a certain integral equation, the integral extending from 0 to ∞, it has been evident that the problem of the cylindrical antenna (with tube-shaped ends) can also be solved mathematically exact. In my book (Hallén 1953, pp. 426-428) I have also given the exact solution of the problem of the first reflected wave, as it is derived from the linearized antenna integral equation (5a), when the distance ℓ from the end to the feeding point tends to infinity. It is also quite certain that a finite ℓ makes no real extra difficulties and that the following waves can be exactly determined as well, although formulas become more involved in these cases. It is our object in this paper to make a corresponding investigation based on the exact integral equation (3) for a tube-shaped cylindrical antenna, i. e. to obtain the exact expression for the first reflected wave, and to compare the result with the corresponding result from the linearized equation. In doing so we will get an exact knowledge of the particular role played by the antenna ends. Of course the tube-shaped ends are only one of several possibilities, but for the role of the ends the tube end can serve as a type. As has been expected by me--I do not answer for others--the difference between the results from the exact equation (3) and the linearized equation (5a) consists in terms, which have the small factor $a\beta$. For thin antennas in fact the linearized equation is the most general, it is independent of the particular form of the ends, and what is common to all antennas with differently shaped end-surfaces is just what the linearized equation gives. But is is very important to know, numerically, for a special case how far the agreement between the exact solution and the solution of the linearized equation extends.

Two features can be expected to be of special interest. When the reflected travelling current wave has travelled a considerable distance from the end it is obviously of very little concern whether we have used the exact or the linearized equation. But in the beginning and especially in the very end-point of the antenna the difference must be expected to be the greatest. Therefore we will determine the exact end admittance of the reflected wave as it is derived from (3) and compare it with the end admittance derived from (5) which has already been determined (Hallén 1953, diagram fig. 412). Further the open tube-end will carry a small internal current, which analogue to the inner current of the outgoing wave can be expected to be aperiodical in space and only represent an end-charging current. This mainly capacitive current branches off from the reflected wave current in the end of the antenna. We will determine it numerically in the end-point. This current has no correspondence in the linearized solution, where there is, however, just as was the case with the outgoing wave, a fictitious small, slow, extra wave current without physical significance (Hallén 1953, p. 431). Besides this, the exact shape of the field round the antenna end will be of interest.

For the deduction from (5a) of the complete system of integral equations, one for each travelling current wave, see (Hallén 1953, pp. 421-425, 433-434). Although the main feature of the reasoning is based only on the true antenna waves, it can be repeated in exactly the same way even if we start with the exact standing wave equation (3), in spite of the fact that it's general solution contains even the aperiodic tube waves. In this paper we will, however, limit the investigation to the first reflected wave, and, for simplicity, assume the antenna length 2ℓ to be very great. The influence of ℓ on the field and current conditions around the end apparently is very small anyway. Then we do not need the whole system of integral equations and can obtain the needed one in a simple way.

We introduce in our equations the distance z along the antenna from the end and thus have $x = z - \ell$, where $z = 0$, when $x = - \ell$. We usually need not extend z to greater values than $x < \ell$ and (for this part of the antenna) have $|x| = \ell - z$. The incoming current wave according to (9) is in our new coordinate:

$$I(z) = -\frac{4\pi}{Z_0} V_0 \frac{j\beta}{2\pi} \int_{\Gamma} \frac{e^{ja(z - \ell)}}{(a^2 - \beta^2)\, I_0\left(a\sqrt{a^2 - \beta^2}\right) K_0\left(a\sqrt{a^2 - \beta^2}\right)}\, da \tag{22}$$

If ℓ increases infinitely there will be no reflected wave coming from the farther end $z = 2\ell$, but only, besides the incoming current $I(z)$ and corresponding potential wave $V_0\, e^{-j\beta|\ell - z|}$, one reflected current wave $i(z)$ and a corresponding potential wave $V_1\, e^{-j\beta z}$. Thus we get from (3), when $\ell \longrightarrow \infty$ the integral equation:

$$\int_0^{\infty} \left[I(\zeta) + i(\zeta)\right] d\zeta \,\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\beta r}}{r}\, d\varphi = \frac{4\pi}{Z_0}\left(-V_0\, e^{-j\beta|\ell - z|} + V_1\, e^{-j\beta z}\right) \tag{23}$$

where $\zeta = \xi + \ell$ and $r = \sqrt{(z - \xi)^2 + 4a^2 \sin^2 \frac{1}{2}\varphi}$. Now, if we fictively define $I(\zeta)$ as before even for negative ζ, we have according to (4):

$$\int_{-\infty}^{\infty} I(\zeta)\, d\zeta \,\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\beta r}}{r}\, d\varphi = -\frac{4\pi}{Z_0} V_0\, e^{-j\beta|\ell - z|}$$

Subtracting this equation from eqn. (23) we get the <u>integral equation for the first reflected current wave</u> (cf. Hallén 1953, p. 426):

$$\int_{-\infty}^{\infty} i(\zeta)\, d\zeta \,\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\beta r}}{r}\, d\varphi = \begin{cases} \dfrac{4\pi}{Z_0} V_1\, e^{-j\beta z}, & z > 0 \\[2ex] g(z), & z < 0 \end{cases} \tag{24}$$

where $i(z)$ is, for $z > 0$, the unknown reflected current wave, and, by arbitrary definition, for $z < 0$:

$$i(z) = - I(z) \tag{25}$$

and $g(z)$ an unknown function which we need not determine. V_1 is a potential amplitude, which has to be determined. Eqn. (24) is valid whether the antenna length is infinite or finite. Now we will use the simplification, which comes from making $\ell = \infty$. The only consequence of this is that the incoming current (22) will be simplified. In fact, the expression (22) for the incoming current wave approaches, for big ℓ :

$$I(z) = - \frac{4\pi}{Z_0} V_0 \, e^{j\beta(z - \ell)} \Omega_{\ell-z}^{-1} \tag{26}$$

where

$$\Omega_{\ell-z} = \log \frac{2(\ell - z)}{a^2 \beta} - \gamma - j\frac{\pi}{2} + \mathcal{L}\,(2\beta(\ell-z)) \tag{27}$$

In (26) terms of the order of magnitude $\Omega_{\ell-z}^{-3}$ have been neglected. Neglecting even z beside ℓ we get a constant amplitude of the incoming current wave by substituting Ω_ℓ for $\Omega_{\ell-z}$, where $\Omega_\ell = \log \frac{2\ell}{a^2\beta} - \gamma - j\frac{\pi}{2}$. The corresponding asymptotic expressions for the incoming wave of the vector potential and the scalar potential as derived from (15a, 16a) are (with x substituted by $z - \ell$):

$$A_z = - \frac{V_0}{c} e^{j\beta(z - \ell)} - \frac{V_0}{c} 2 \log \frac{a}{\rho} e^{j\beta(z - \ell)} \Omega_{\ell-z}^{-1} \qquad \rho > a \tag{28}$$

$$V = V_0 \, e^{j\beta(z - \ell)} + V_0 \, 2 \log \frac{a}{\rho} e^{j\beta(z - \ell)} \Omega_{\ell-z}^{-1} \qquad \rho \quad a \tag{29}$$

where ℓ is very big in comparison to z, ρ, a. For $\rho < a$ we simply get from (15b) and (16b) the asymptotic values:

$$A_x = - \frac{V_0}{c} e^{j\beta(z - \ell)} \tag{30}$$

$$V = V_o\, e^{j\beta(z - \ell)} \tag{31}$$

and in this case the approach is closer; the neglected terms are small essentially as $e^{-\frac{\ell}{a}}$.

The first terms in (28, 29) together with (30, 31) form a potential "field" of constant amplitude and constant direction. It has no physical content because the field strengths $\underline{B}$ and $\underline{E}$ as derived from it are zero everywhere. The electromagnetic potentials of a radiation field are more purely mathematical and less physical than in quasi-stationary cases. It is therefore more adequate to use the incoming current wave amplitude as a measure of the incoming wave than the potential (which we keep only in the unimportant first terms of (28 - 31)). We put the constant

$$\underline{I} = \frac{4\pi}{Z_o} V_o\, e^{-j\beta\ell}\, \Omega_\ell^{-1} \tag{32}$$

and have for the incoming wave the asymptotic formulas:

$$I(z) = -\underline{I}\, e^{j\beta z} \tag{33}$$

$$A_z = -\frac{V_o}{c} e^{j\beta(z - \ell)} - \frac{\mu_o \underline{I}}{2\pi} \log \frac{a}{\rho}\, e^{j\beta z} \quad , \quad \rho > a \tag{34a}$$

$$A_z = -\frac{V_o}{c} e^{j\beta(z - \ell)} \quad , \quad \rho < a \tag{34b}$$

$$V = V_o\, e^{j\beta(z - \ell)} + \frac{Z_o \underline{I}}{2\pi} \log \frac{a}{\rho}\, e^{j\beta z} \quad , \quad \rho > a \tag{35a}$$

$$V = V_o\, e^{j\beta(z - \ell)} \quad , \quad \rho < a \tag{35b}$$

$$B\varphi = -\frac{\mu_o \underline{I}}{2\pi\rho} e^{j\beta z} \quad , \quad \rho > a \tag{36a}$$

$$B\varphi = 0 \quad , \quad \rho < a \tag{36b}$$

The expressions (34-36) have an independent physical sense only for $z > 0$. For $z < 0$, i. e. outside the antenna end, one cannot reasonably make a difference between incoming and reflected wave (or this difference would be artificial). In that case (34-36) represent a first part of a total wave (see further below).

SOLUTION OF THE INTEGRAL EQUATION FOR THE REFLECTED CURRENT WAVE

We now introduce (33) in (25) and (24) and turn to the exact solution of the integral equation (24). The procedure for solving (24) is exactly the same as that which I have carried out before (Hallén 1953, p. 426-430) for the corresponding linearized equation. The only difference is that we have a different kernel, which now is the exact one for a tube-shaped antenna. The mathematical method is in its main feature that of Hopf and Wiener (see Titchmarsh 1937, p. 339) and it was first used on a physical tube problem (an accoustical one) by (Levine and Schwinger 1948). Our procedure is, however, entirely different from that of the latter. One of the advantages we get is that we never need a study of the field to get boundary conditions, but that these are automatically filled and all constants known. (See the very simple deduction of the formulas (48-50) below). In fact, we need not know the formulas for the potentials and fields (15-19, 34-36, etc.) for solving the problem; however, we get them, as we have seen, very easily and they are of interest in themselves.

In order to facilitate our computation we temporarily assume that β has a small negative imaginary term $\beta = \beta_1 - j\varepsilon$ so that β is situated in the lower complex semi-plane. We dissolve the kernel of the integral equation (24) into a Fourier integral (Hallén 1953, eqn. 35.22):

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-j\beta r}}{r}\, d\varphi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\alpha(z-\zeta)}\, 2\, I_o\left(a\sqrt{\alpha^2-\beta^2}\right) K_o\left(a\sqrt{\alpha^2-\beta^2}\right) d\alpha$$

Even $i(\zeta)$ is expressed as a Fourier integral

$$i(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\alpha\zeta}\left[F_+(\alpha) + F_-(\alpha)\right] d\alpha \tag{37}$$

where $F_+(\alpha) = \int_0^{\infty} i(u)\, e^{-j\alpha u}\, du$, $F_-(\alpha) = \int_{-\infty}^{0} i(u)\, e^{-j\alpha u}\, du$. $F_+(\alpha)$ is unknown but regular in the whole <u>lower</u> complex α-semi-plane. When α approaches $-j\infty$ we get the limit value of $F_+(\alpha)$:

$$F_+(\alpha) \rightarrow i(0)\int_0^{\infty} e^{-j\alpha u}\, du = \frac{i(0)}{j\alpha} \tag{38}$$

$F_-(\alpha)$ is known because for negative u , according to (25, 33), we have $i(u) = \underline{I}\, e^{j\beta u}$ and thus:

$$F_-(\alpha) = \underline{I}\int_{\infty}^{0} e^{j(\beta-\alpha)u}\, du = \frac{\underline{I}}{j(\beta-\alpha)} \tag{39}$$

which is a regular function of α in the whole <u>upper</u> complex α-semi-plane.
We finally dissolve even the right side of (24) and find it to be:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\alpha z}\left[G_+(\alpha) + G_-(\alpha)\right] d\alpha$$

where $G_+(\alpha)$ is known:

$$G_+(\alpha) = \int_0^{\infty} \frac{4\pi}{Z_o} V_1\, e^{-j\beta u}\, e^{-j\alpha u}\, du = \frac{4\pi}{Z_o}\frac{V_1}{j(\beta+\alpha)} \tag{40}$$

which is a regular function of a in the whole lower complex a-semi-plane. $G_-(a)$ is unknown:

$$G_-(a) = \int_{-\infty}^{0} g(u)\, e^{-jau}\, du$$

but regular in the upper complex a-semi-plane. It disappears in infinity essentially as $\frac{1}{-ja}$ in this semi-plane.

Introducing the Fourier integrals in (24) we get:

$$\int_{-\infty}^{\infty} d\xi \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\gamma\xi}\left[F_+(\gamma) + F_-(\gamma)\right] d\gamma \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ja(z-\xi)}\, 2I_o\left(a\sqrt{a^2-\beta^2}\right) K_o\left(a\sqrt{a^2-\beta^2}\right) da =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jaz}\left[G_+(a) + G_-(a)\right] da$$

Using again Fourier's integral theorem in the opposite direction the integrations in ξ and in γ gives us back $\left[F_+(a) + F_-(a)\right]$ and our integral equation attains the final form:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jaz} \left\{ 2I_o\left(a\sqrt{a^2-\beta^2}\right) K_o\left(a\sqrt{a^2-\beta^2}\right)\left[F_+(a) + F_-(a)\right] - G_+(a) - G_-(a) \right\} da = 0 \qquad (41)$$

Following the Hopf-Wiener method we write:

$$2\, I_o\left(a\sqrt{a^2 - \beta^2}\right) K_o\left(a\sqrt{a^2 - \beta^2}\right) = \frac{\varphi_1(a)}{\varphi_2(a)} \qquad (42)$$

where

$$\varphi_1(a) = \exp \frac{1}{2\pi j} \int_{-\infty-j\delta}^{\infty-j\delta} \frac{d\xi}{\xi-a} \log\left[2I_o\left(a\sqrt{\xi^2-\beta^2}\right) K_o\left(a\sqrt{\xi^2-\beta^2}\right)\right] \qquad (43)$$

$$\varphi_2(\alpha) = \exp \frac{1}{2\pi j} \int_{-\infty+j\delta}^{\infty+j\delta} \frac{d\zeta}{\zeta-\alpha} \log \left[2\, I_o(a\sqrt{\zeta^2-\beta^2})\, K_o\, (a\sqrt{\zeta^2-\beta^2})\right] \tag{44}$$

Here δ is a positive number so small that $\delta < \varepsilon$, i. e. both the integration paths in (43, 44) go between $\beta = \beta_1 - j\varepsilon$ and $-\beta = -\beta_1 + j\varepsilon$. Then $\varphi_1(\alpha)$ is regular and different from zero in the whole upper complex α-semi-plane, $\varphi_2(\alpha)$ in the lower semi-plane, and both these regions can be extended somewhat to the other side of the real axis provided that the border line remains within the strip $\pm\,\delta$. We even find

$$\varphi_1(-\alpha)\, \varphi_2\, (\alpha) = 1 \tag{45}$$

The equation (41) now takes the form:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\alpha z}\, \frac{\varphi_1(\alpha)}{\alpha^2 - \beta^2} \left\{ \frac{\alpha^2-\beta^2}{\varphi_2(\alpha)} \left[F_+(\alpha) + F_-(\alpha)\right] - \frac{\alpha^2-\beta^2}{\varphi_1(\alpha)}\left[G_+(\alpha) + G_-(\alpha)\right] \right\} d\alpha = 0 \tag{46}$$

Here $\dfrac{\varphi_1(\alpha)}{\alpha^2-\beta^2}$, $\dfrac{\alpha^2-\beta^2}{\varphi_2(\alpha)}$, $\dfrac{\alpha^2-\beta^2}{\varphi_1(\alpha)}$, $F_-(\alpha)$ and $G_+(\alpha)$ all remain analytical within the strip $\pm\,\delta$. If we anticipate, as is natural, that when β is slightly complex $(\beta = \beta_1 - j\varepsilon)$, $i(z)$ decreases at least as $e^{-\varepsilon z}$ $(z > 0)$, and $g(z)$ at least as $e^{\varepsilon z}$ $(z < 0)$, then even $F_+(\alpha)$ and $G_-(\alpha)$ will remain analytic within the strip. The integration path in (46) thus can be arbitrary if it remains within the strip. The necessary conditions for this is that the bracket expression vanishes and therefore with respect to (39, 40):

$$P(\alpha) = \frac{\alpha^2-\beta^2}{\varphi_2(\alpha)}\, F_+(\alpha) + j\, \frac{\alpha+\beta}{\varphi_2(\alpha)}\, I = -\, j\, \frac{\alpha-\beta}{\varphi_1(\alpha)}\, \frac{4\pi V_1}{Z_o} + \frac{\alpha^2-\beta^2}{\varphi_1(\alpha)}\, G_-(\alpha) \tag{47}$$

The left side is analytic in the lower semi-plane, the right side in the upper semi-plane; within the strip both sides are equal. Consequently they represent the same analytic function. As F_+ and G_- are limited, $P(\alpha)$

must be a polynomial of no higher than the second degree and as F_+ and G_- vanish in infinity it cannot even be of more than the first degree. We may thus put

$$P(\alpha) = C_1 + C_2\,(\alpha - \beta)$$

where C_1 and C_2 are two constants.

Putting $\alpha = \beta$ we get from the left side of (47)

$$C_1 = j\,\frac{2\beta}{\varphi_2(\beta)}\,I$$

If we divide (47) by $(\alpha - \beta)$ and let α go towards $-j\,\infty$, we get with respect to (38):

$$C_2 = \frac{\alpha + \beta}{j\alpha\,\varphi_2(\alpha)}\,i(0) + j\,\frac{\alpha + \beta}{(\alpha-\beta)\,\varphi_2(\alpha)}\,I = j\,\frac{I - i(0)}{\varphi_2(\alpha)}$$

and thus according to (25, 33):

$$C_2 = 0$$

Hence P is a constant:

$$P = j\,\frac{2\beta}{\varphi_2(\beta)}\,I \tag{48}$$

We determine even V_1 by using the right side in (47) where we put $\alpha = -\beta$, which gives:

$$P = j\,\frac{2\beta}{\varphi_1(-\beta)}\;\frac{4\pi\,V_1}{Z_o}$$

or with the help of (45) and (48)

$$V_1 = \frac{Z_o}{4\pi\,\varphi_2^{\,2}(\beta)}\,I \tag{49}$$

We can even introduce the <u>end admittance of the reflected wave</u> (when the incoming wave comes from infinity):

$$Y_\infty = \frac{I}{V_1} = \frac{4\pi}{Z_o}\varphi_2^2\ (\beta) \tag{50}$$

(Numerical values see Fig. 1)

The Fourier transform of the unknown function $i(z)$ is now known, for from (47, 48) we get

$$F_+(a) + F_-(a) = j\,\frac{2\beta}{\varphi_2(\beta)}\,I\,\frac{\varphi_2(a)}{a^2 - \beta^2} \tag{51}$$

From (37) we now find the solution of the problem. The reflected current wave is:

$$i(z) = I\,\frac{j\beta}{\pi\,\varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{e^{jaz}\,\varphi_2(a)}{a^2 - \beta^2}\,da \tag{52a}$$

or with respect to (42):

$$i(z) = I\,\frac{j\beta}{2\pi\,\varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{e^{jaz}\,\varphi_1(a)}{(a^2-\beta^2)\ I_o\ (a\sqrt{a^2-\beta^2})\ K_o\ (a\sqrt{a^2-\beta^2})}\,da \tag{52b}$$

The factor before the integral in (52a) can also be written $V_1\,\dfrac{j4\beta\,\varphi_2(\beta)}{Z_o}$. The properties of $\varphi_2(a)$, $i(z)$ and $g(z)$ will be studied in the following.

We state that all the equations (37-40), (45-52a) are exactly the same as when the equation was linearized (Hallén 1953, eqn. 35-52 - 35-57). Thus (52a, b) even represent the reflected current wave, as derived from the linearized equation, but with φ_1 and φ_2 defined as in (43, 44) with the difference that the bracket $[2I_oK_o]$ is substituted by a logarithmic expression, which is the limit value of the Bessel function expression if powers of $a\beta$ are suppressed. Thus there is exactly the same relation between our solution to the reflected wave integral equation (24) and the corresponding one of the linearized equation as there is between the two exact solutions (9) and (10) of the exact integral equation (4) and the linearized one (6) for the outgoing wave. It is however our task to carry out the comparison completely numerically and find out how thick the antenna may be if the linearization shall still be permitted.

It is easy to check the result (52a). For $z < 0$ we can add an infinitely big circle in the lower complex α-semi-plane to the integration path. As $\varphi_2(\alpha)$ is regular in the lower half-plane, the integrand has no other singularity within the contour than the pole $\alpha = \beta = \beta_1 - j\varepsilon$ and we get for negative z the value

$$i(z) = I \frac{j\beta}{\pi\, \varphi_2(\beta)} (-2\pi j) \frac{\varphi_2(\beta)}{2\beta} e^{j\beta z} = I\, e^{j\beta z} \quad , \quad z < 0$$

in agreement with (25, 33). This is the fictive current which we have introduced to compensate the fictive continuation of the incoming current wave:

$$i(z) + I(z) = 0 \quad , \quad z < 0$$

For $z > 0$ we introduce (52b) in the left side of (24) and find this side of (24) to be, with respect to (14):

$$I \frac{j\beta}{\pi\, \varphi_2(\beta)} \int_{-\infty}^{\infty} \frac{e^{j\alpha z}\, \varphi_1(\alpha)}{\alpha^2 - \beta^2}\, d\alpha \tag{53}$$

and (for positive z) we close the integration path in the upper semi-plane, where $\varphi_1(\alpha)$ is regular. The only singularity within the contour now is the pole $\alpha = -\beta = -\beta_1 + j\varepsilon$ and the expression (53), on account of (45, 49), becomes

$$I \frac{\varphi_1(-\beta)}{\varphi_2(\beta)} e^{-j\beta z} = \frac{I}{\varphi_2^2(\beta)} e^{-j\beta z} = \frac{4\pi}{Z_0} V_1\, e^{-j\beta z} \quad , \quad z > 0$$

in accordance with the right side of (24), which equation thus is fulfilled.

For negative values of z the expression (53) represents the hitherto unknown function $g(z)$ of eqn. (24).

The reflected current wave (52), with $z > 0$, consists (as was the case with the outgoing current (9)) of two parts: an outer antenna wave i_e and an inner wave guide wave i_i, below cutoff frequency, which penetrates into the tube from the end but which decreases exponentially. For thick antennas the latter is not quite unessential as it represents a current which branches off from the main antenna current wave. It is mainly a charge current, and if the antenna has, for instance, flat ends instead of tube-shaped ones, it corresponds to the small current which charges and discharges the end surface. We can easily split (52b) into these two currents. We observe that

$$\frac{1}{I_o(a\sqrt{\alpha^2-\beta^2})\,K_o(a\sqrt{\alpha^2-\beta^2})} = a\sqrt{\alpha^2-\beta^2}\left[\frac{I_1}{I_o}+\frac{K_1}{K_o}\right]_{a\sqrt{\alpha^2-\beta^2}} \tag{54}$$

and

$$i(z) = i_e(z) + i_i(z)$$

where

$$i_e(z) = I\,\frac{j\alpha\beta}{2\pi\,\varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{e^{j\alpha z}\,\varphi_1(\alpha)}{\sqrt{\alpha^2-\beta^2}}\cdot\frac{K_1\,(a\sqrt{\alpha^2-\beta^2})}{K_o\,(a\sqrt{\alpha^2-\beta^2})}\,,\quad z>0 \tag{55}$$

$$i_i(z) = I\,\frac{j\alpha\beta}{2\pi\,\varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{e^{j\alpha z}\,\varphi_1(\alpha)}{\sqrt{\alpha^2-\beta^2}}\cdot\frac{I_1\,(a\sqrt{\alpha^2-\beta^2})}{I_o\,(a\sqrt{\alpha^2-\beta^2})}\,,\quad z>0 \tag{56a}$$

With residue calculus we find the latter expression to be the following series:

$$i_i(z) = I\,\frac{j\alpha\beta}{\varphi_2(\beta)}\sum_{1}^{\infty}\frac{e^{-\sqrt{\xi_m^2-a^2\beta^2}\,\frac{z}{a}}\;\varphi_1\!\left(\frac{j}{a}\sqrt{\xi_m^2-a^2\beta^2}\right)}{\sqrt{\xi_m^2-a^2\beta^2}} \tag{56b}$$

where the ξ_m are the zeros of the Bessel function $J_0(\xi)$. (56b) is a series of wave guide waves (below cutoff frequency) to which we will return later. For a corresponding series expression for the wave guide waves of the outgoing antenna see (Hallén 1953, eqn. 35.25). The linearized equation does not give these waves (56b).

THE ELECTROMAGNETIC FIELD AROUND THE ANTENNA END

We are now in the position that we can give an exact description in every detail of the field around the antenna end (if this is tube-shaped). The way which is physically most straight forward would be to use the true currents, the incoming current (33) and the reflected current (52), for $z > 0$ and express the vector potential from them in the form of an integral extending from $z = 0$ to $z = \infty$ (cf. eqn. (23)). But it is easier, and equally correct, to extend both currents to $z = -\infty$, with the provision (25) that $i(z) + I(z) = 0$ for $z < 0$. The field from the (extended) incoming current $I(z)$ we have already in (34a, b), (35a, b), (36a, b). The vector potential from the (extended) reflected current (52a) is at an arbitrary point ρ , 0 , z :

$$A_z = \frac{\mu_0}{4\pi} I \frac{j\beta}{\pi \varphi_2(\beta)} \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} \frac{e^{-j\beta r}}{r} d\xi \int_{-\infty}^{\infty} \frac{e^{j\alpha\xi} \varphi_2(\alpha)}{\alpha^2 - \beta^2} d\alpha$$

where $r = \left[(z-\xi)^2 + \rho^2 + a^2 - 2a\rho \cos\varphi\right]^{\frac{1}{2}}$. Using (14, 42) this gives

$$A_z = I \frac{\mu_0 \, j\beta}{4\pi^2 \varphi_2(\beta)} \int_{-\infty}^{\infty} \frac{e^{j\alpha z} \varphi_1(\alpha)}{\alpha^2 - \beta^2} \cdot \frac{K_0 \left(\rho\sqrt{\alpha^2-\beta^2}\right)}{K_0 \left(a\sqrt{\alpha^2-\beta^2}\right)} d\alpha \quad , \quad \rho > a \tag{57a}$$

$$A_z = I \frac{\mu_o\, j\beta}{4\pi^2\, \varphi_2(\beta)} \int_{-\infty}^{\infty} \frac{e^{j\alpha z}\, \varphi_1(\alpha)}{\alpha^2 - \beta^2} \cdot \frac{I_o\, (\rho\sqrt{\alpha^2-\beta^2})}{I_o\, (a\sqrt{\alpha^2-\beta^2})}\, d\alpha \quad , \quad \rho < a \tag{57b}$$

For $z > 0$ this represents the reflected vector potential wave, whereas the incoming vector potential wave is given by (34a, b). For $z < 0$, i. e. outside the antenna end we can no longer distinguish between incoming and reflected waves but we get the <u>total</u> vector potential wave by adding (34a, b). In this region $(z < 0)$ it is better to go back to the function φ_2 instead of φ_1. We thus get the <u>total</u> vector potential outside the antenna end:

$$A_z = -\frac{V_o}{c} e^{j\beta(z-\ell)} - I \frac{\mu_o}{2\pi} \log \frac{a}{\rho} e^{j\beta z} +$$

$$+ I \frac{\mu_o\, j\beta}{2\pi^2\, \varphi_2(\beta)} \int_{-\infty}^{\infty} \frac{e^{j\alpha z}\, \varphi_2(\alpha)}{\alpha^2 - \beta^2}\, I_o\, (a\sqrt{\alpha^2-\beta^2})\, K_o\, (\rho\sqrt{\alpha^2-\beta^2})\, d\alpha \quad , \quad \rho > a \tag{58a}$$

$$A_z = -\frac{V_o}{c} e^{j\beta(z-\ell)} + I \frac{\mu_o\, j\beta}{2\pi^2\, \varphi_2(\beta)} \int_{-\infty}^{\infty} \frac{e^{j\alpha z}\, \varphi_2(\alpha)}{\alpha^2 - \beta^2}\, K_o\, (a\sqrt{\alpha^2-\beta^2})\, I_o\, (\rho\sqrt{\alpha^2-\beta^2})\, d\alpha \, , \rho < a \tag{58b}$$

The expressions (58a) and (58b) are in themselves valid even for $z > 0$, although we prefer to use φ_1 instead of φ_2 when $z > 0$. The reason why we always make this distinction, using φ_1 when z is positive but φ_2 when it is negative, is that we can deform the path of integration in the upper half plane in the first case, and in the lower one in the second. It can be shown (cf. 64, 65a) from the integrals in (55) and (57a) that they respresent, for $z > 0$, waves travelling in the direction of positive z (i. e. have a periodic factor $e^{-j\beta z}$), whereas (52a) for $z < 0$ represents waves travelling in the negative direction (periodic factor $e^{j\beta z}$).

From the vector potential we get the magnetic field $B_\varphi = -\frac{\partial A_z}{\partial \rho}$ and thus, using (58a, b) the <u>total</u> magnetic field outside the antenna end $(z < 0)$

becomes:

$$B_\varphi = -\underline{I}\frac{\mu_0}{2\pi\rho}e^{j\beta z} + \underline{I}\frac{\mu_0 j\beta}{2\pi^2 \varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{e^{j\alpha z}\varphi_2(\alpha)}{\sqrt{\alpha^2-\beta^2}} I_o(a\sqrt{\alpha^2-\beta^2})K_1(\rho\sqrt{\alpha^2-\beta^2})\,d\alpha \quad , \quad \rho > a \tag{59a}$$

$$B_\varphi = -\underline{I}\frac{\mu_0 j\beta}{2\pi^2 \varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{e^{j\alpha z}\varphi_2(\alpha)}{\sqrt{\alpha^2-\beta^2}} K_o(a\sqrt{\alpha^2-\beta^2})I_1(\rho\sqrt{\alpha^2-\beta^2})\,d\alpha \quad , \quad \rho < a \tag{59b}$$

These expressions for the total magnetic field are still valid even for $z > 0$, but in this case we should write them (using 42) as:

$$B_\varphi = -\underline{I}\frac{\mu_0}{2\pi\rho}e^{j\beta z} + \underline{I}\frac{\mu_0 j\beta}{4\pi^2 \varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{e^{j\alpha z}\varphi_1(\alpha)}{\sqrt{\alpha^2-\beta^2}} \cdot \frac{K_1(\rho\sqrt{\alpha^2-\beta^2})}{K_o(a\sqrt{\alpha^2-\beta^2})}\,d\alpha \quad , \quad \rho > a \tag{59c}$$

$$B_\varphi = -\underline{I}\frac{\mu_0 j\beta}{4\pi^2 \varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{e^{j\alpha z}\varphi_1(\alpha)}{\sqrt{\alpha^2-\beta^2}} \cdot \frac{I_1(\rho\sqrt{\alpha^2-\beta^2})}{I_o(a\sqrt{\alpha^2-\beta^2})}\,d\alpha \quad , \quad \rho < a \tag{59d}$$

In the term outside the integral we recognize the field of the incoming wave (36). If we put $\rho = a$, we find from (59d) the limit value of $-\frac{2\pi a}{\mu_0}B_\varphi(a)$ to be equal to the inner current (56a), and from (59c) the outer limit value of $+\frac{2\pi a}{\mu_0}B_\varphi(a)$ to be equal to the sum of the incoming current (33) and the reflected outer current (55). Our interpretation of the physical significance of the two terms (55) and (56a) of the reflected current is thus justified.

It is easy to verify that outside the antenna end the magnetic field is continuous at $\rho = a$. If we subtract (59b) from (59a) and put $\rho = a$ we find from (Watson 1944, p. 80, eqn. 20) the result zero. However, it is a lack of elegance not to have a common formula for $\rho < a$ as well as for $\rho > a$ outside the antenna end, because the limit $\rho = a$ between the two regions has no physical significance. We find a common formula by deforming the integration path in (59a) and (59b) into a double line in the lower complex

α-semi-plane from $\alpha = \beta$ to $\alpha = \beta - j\,\infty$. If we further substitute $\sqrt{\alpha^2-\beta^2} = -\,j\beta u$ we get, for $z < 0$, the common formula:

$$B_\varphi = -\,I\,\frac{\mu_0\, j\beta}{2\pi\,\varphi_2(\beta)}\int_0^\infty \frac{e^{\beta z\sqrt{u^2-1}}\,\varphi_2(-j\beta\sqrt{u^2-1})}{\sqrt{u^2-1}}\,J_0\,(a\beta u)\,J_1\,(\rho\beta u)\,du \qquad (59e)$$

valid for all ρ but only for negative z. The integration path should avoid the singular point $u = 1$ in the upper complex semi-plane.

We finally give even the components of the electric field which are equally easy to deduce. The <u>total</u> electric field is:

$$E_z = -\,I\,\frac{Z_0}{4\pi^2\,\varphi_2(\beta)}\int_{-\infty}^{\infty} e^{j\alpha z}\,\varphi_1(\alpha)\,\frac{K_0(\rho\sqrt{\alpha^2-\beta^2})}{K_0(a\sqrt{\alpha^2-\beta^2})}\,d\alpha \quad , \quad \rho > a \qquad (60a)$$

$$E_z = -\,I\,\frac{Z_0}{4\pi^2\,\varphi_2(\beta)}\int_{-\infty}^{\infty} e^{j\alpha z}\,\varphi_1(\alpha)\,\frac{I_0(\rho\sqrt{\alpha^2-\beta^2})}{I_0(a\sqrt{\alpha^2-\beta^2})}\,d\alpha \quad , \quad \rho < a \qquad (60b)$$

$$E_\rho = I\,\frac{Z_0}{2\pi\rho}\,e^{j\beta z} - I\,\frac{Z_0\,j}{4\pi^2\,\varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{\alpha\,e^{j\alpha z}\,\varphi_1(\alpha)}{\sqrt{\alpha^2-\beta^2}}\,\frac{K_1(\rho\sqrt{\alpha^2-\beta^2})}{K_0(a\sqrt{\alpha^2-\beta^2})}\,d\alpha \quad , \quad \rho > a \qquad (61a)$$

$$E_\rho = I\,\frac{Z_0\,j}{4\pi^2\,\varphi_2(\beta)}\int_{-\infty}^{\infty}\frac{\alpha\,e^{j\alpha z}\,\varphi_1(\alpha)}{\sqrt{\alpha^2-\beta^2}}\,\frac{I_1(\rho\sqrt{\alpha^2-\beta^2})}{I_0(a\sqrt{\alpha^2-\beta^2})}\,d\alpha \quad , \quad \rho < a \qquad (61b)$$

The first term of (61a) is the incoming wave (which has no z-component). The equations (60-61) are valid both for positive and negative values of z, but for negative z it is better to write the common formulas for the total field outside the antenna end:

$$E_z = - I \frac{Z_0 \beta}{2\pi \varphi_2(\beta)} \int_0^\infty \frac{u e^{\beta z \sqrt{u^2-1}}}{\sqrt{u^2 - 1}} \varphi_2 (-j\beta \sqrt{u^2-1}) J_o (a\beta u) J_o (\rho\beta u) du \tag{60c}$$

$$E_\rho = I \frac{Z_0 \beta}{2\pi \varphi_2(\beta)} \int_0^\infty e^{\beta z \sqrt{u^2-1}} \varphi_2(-j\beta \sqrt{u^2-1}) J_o (a\beta u) J_1 (\rho\beta u) du \tag{61c}$$

where the integration path avoids the point $u = 1$ in the upper complex u-semi-plane. The formulas (60c, 61c) are valid for all ρ but only for negative z.

Formulas (52-61) give a complete mathematical solution to the problem under investigation.

INVESTIGATION OF THE ANTENNA CURRENT

The inner current (56b) is aperiodic and decreases exponentially. Most interest has its initial value at the antenna end:

$$i_1(0) = I \frac{ja\beta}{\varphi_2(\beta)} \sum_1^\infty \frac{\varphi_1(\frac{j}{a}\sqrt{\xi_m^2 - a^2\beta^2})}{\sqrt{\xi_m^2 - a^2\beta^2}} \tag{62}$$

to which we will return later. It constitutes a branch-off mainly capacitive current at the end. With the help of (49) we may define an inner end admittance

$$Y_i = \frac{i_1(0)}{V_1} = \frac{4\pi}{Z_o} ja\beta \; \varphi_2(\beta) \sum_1^\infty \frac{\varphi_1(\frac{j}{a}\sqrt{\xi_m^2 - a^2\beta^2})}{\sqrt{\xi_m^2 - a^2\beta^2}} \tag{63}$$

The outer reflected current (55) is periodic and represents a wave; by changing the integration path through the upper complex α-semi-plane we can write it in the form

$$i_e(z) = \frac{I}{\varphi_2^{\,2}(\beta)} \mathcal{V}_\infty\ (\beta z)\ e^{-j\beta z} = \frac{4\pi}{Z_o} V_1 \mathcal{V}_\infty\ (\beta z)\ e^{-j\beta z} \tag{64}$$

where

$$\mathcal{V}_\infty (\beta z) = -\frac{1}{2}\beta\, \varphi_2(\beta) \int_{-\beta}^{-\infty} \frac{e^{j(\alpha + \beta) z}\, \varphi_1\ (\alpha)\ d\alpha}{(\alpha^2 - \beta^2)\ K_o\ (a\sqrt{\alpha^2-\beta^2})\ K_o\ (a\sqrt{\alpha^2-\beta^2}\ e^{j\pi})} \tag{65a}$$

(here $\sqrt{\alpha^2 - \beta^2}$ is positive along the integration path).

By a suitable change of variable this also becomes:

$$\mathcal{V}_\infty\ (\beta z) = \varphi_2(\beta)\ \frac{2j}{\pi^2} \int_0^\infty \frac{e^{\beta z(j - \sqrt{u^2-1})}\ \varphi_1(j\beta\sqrt{u^2-1})}{u\sqrt{u^2-1}\ H_o^{(1)}\ (a\beta u)\ H_o^{(2)}\ (a\beta u)}\ du \tag{65b}$$

where u shall avoid the point $u = 1$ in the upper complex u-semi-plane. According to (65a) the amplitude function $\mathcal{V}_\infty(\beta z)$ is an aperiodic function of βz. The index ∞ is used in order to remind us of the fact that our reflected wave arises from an incoming current wave which has come from infinity. If it comes from a finite distance ℓ the problem can also be solved but gives a more complicated formula. A finite ℓ, however, influences the reflected wave rather slightly. It should be noted that (65a, b) differ only by the two factors $\varphi_2(\beta)$ and φ_1 from the corresponding formulas for the outgoing wave (Hallén 1953, p. 411-412).

The exact numerical evaluation of $\mathcal{V}_\infty\ (\beta z)$ from (65a) or (65b) for all values of z is unhappily very complicated. One can fall back on series expansions (see further below) which are especially good when z is not too close to 0. For $z = 0$ we have however the value of the reflected current according to (49)

$$i(0) = i_e(0) + i_i(0) = I = \frac{4\pi}{Z_o}\, \varphi_2^{\,2}(\beta)\ V_1 \tag{66}$$

and according to (50) the total end admittance for the reflected wave

$$Y_{\infty} = Y_e + Y_1 = \frac{i(0)}{V_1} = \frac{4\pi}{Z_0} \varphi_2{}^2(\beta) \tag{67}$$

where $\varphi_2(\beta)$ is simpler to determine numerically. That has been done in this investigation for a series of values of $a\beta$, which give the <u>exact values of the end admittance of the straight cylindrical tube-shaped antenna</u>, shown as a diagram in Figure 1 and Figure 2. Instead of $a\beta$ we have plotted the ratio $\frac{\lambda}{a}$ between wave length in free space and radius of cross section. We have the simple relation $\frac{\lambda}{a} = \frac{2\pi}{a\beta}$. Even Y_1 has been numerically computed for a few values and as soon as sufficient computing aid is available to me I will publish a complete diagram. Thus Y_{∞}, Y_e and Y_1 are all known. For details of the numerical determination of φ_2 see below.

Before we proceed further we will even mention the corresponding result as derived from the linearized integral equation. In that case there is no inner current i_1. In fact, all the equations (56b), (62), (63) above contain the small factor $a\beta$, which is supposedly negligible at the linearization. Thus with linearization we have (Hallén 1953, eqn. 35.57):

$$\psi_{\infty}(0) = \varphi_2{}^2(\beta)$$

and

$$Y_{\infty} = Y_e = \frac{4\pi}{Z_0} \varphi_2{}^2(\beta), \quad \text{(linearized equation)} \tag{68}$$

where in this case

$$\varphi_2(\alpha) = \exp \frac{1}{2\pi j} \int_{-\infty + j\delta}^{\infty + j\delta} \frac{d\xi}{\xi - \alpha} \log \log \frac{4}{a^2(\xi^2 - \beta^2)\, \gamma_1{}^2}, \quad \text{(linearized equation)} \tag{69}$$

The function (68) has formerly been computed (Hallén 1953, p. 432, Fig. 412) and it is introduced in our Figure 1 here also for comparison with the exact value for the tube-shaped antenna.

For arbitrary values of z the amplitude function $\Psi_{\infty}(\beta z)$, as it follows from the linearized formula corresponding to (65b) or (65a) (or (64) and (55)), can be expanded into a rapidly decreasing series:

$$\Psi_{\infty}(\beta z) = \Omega_z^{-1} + \Omega_z^{-3}\left[\ell^{11} + \ell^{01} - \ell^2 - \frac{\pi^2}{6}\right]_{2\beta z} +$$

$$+ \Omega_z^{-4}\left[-\ell 3^{111} - 2\ell^{101} - \ell^{011} + \ell\,(3\ell^{11} + 3\ell^{01} - 2\ell^2) +\right.$$

$$\left. + \ell_0\,(\ell^{01} - \ell^{11}) - 2\,S_3\right]_{2\beta z} + \ldots \text{ (linearized equation)} \qquad (70)$$

where $\Omega_z = -2\log a\beta - 2\gamma - j\pi + \ell_0\,(2\beta z) + \ell\,(2\beta z)$ and the different functions ℓ are the amplitude functions of the iterated sine and cosine integrals of different orders as defined in (Hallén 1947, p. 4; Hallén 1948b, p. 4-5; Hallén 1953, p. 435; Hallén 1955, p. 4-6). They are tabulated in (Hallén 1955, Table IV-VI). The parameter Ω_z increases with increasing z and the expressions within the brackets in (70) decrease, with the exception of the constant, essentially as $\frac{1}{\beta z}$. Thus the accuracy of (70) increases with z and it is a very good expression. An exception is the very neighborhood of the starting point (antenna end) $z = 0$, where the series gives

$$\Psi_{\infty}(0) = \Omega_0^{-1} + \frac{\pi^2}{6}\,\Omega_0^{-3} + 4\,S_3\,\Omega_0^{-4} + \ldots \qquad (71)$$

where $S_3 = \sum_1^{\infty}\frac{1}{n^3} = 1.2020567$ and $\Omega_0 = -2\log a\beta - 2\gamma - j\pi$. Even this approximate value (the end admittance derived from it) is entered in Figure 1 for comparison. I point out that generally (70) is much more exact than (71). The antenna current amplitude changes more rapidly in the beginning and there the series is least favorable. Observe also that in (70) and (71) the second negative power of Ω_z , resp Ω_0 is lacking so that the following terms are very small compared to the first one. Series corresponding to (70) for the outgoing wave and for reflected waves when the finite length of the antenna is taken into consideration have also earlier been derived (Hallén 1947, p. 1141 and p. 1145; Hallén 1948b, eqn. 55; Hallén 1953, p. 436).

The series (70) is much more easily derived directly from the linearized integral equation than by series expansion of an exact solution (Hallén 1953, p. 433-436). The only attempt so far to evaluate for all z-values the exact expression (65b) for the wave amplitude of the reflected wave on a tube-shaped antenna has been series expansion, but as it involves the neglect of powers of $a\beta$, this only leads to the already known approximate expression (70).

It is, however, fortunate that at least the end value for $z = 0$ is exactly numerically known (the end admittance) because when summing up the series of travelling waves on a finite antenna, the end admittances play an essential role and the total current is numerically sensitive just to this end value. So, even if at the moment the numerical evaluation of the exact reflected current wave along the whole antenna (from 65b) yet remains, the fact that we know $\gamma_\infty(0)$ exactly is an essential result. Combined with (70) for those parts of the antenna which are not quite close to the end it gives a very exact knowledge of the antenna current everywhere.

For the numerical evaluation of the exact expression for the antenna end admittance (50, 44) one has to transform the integral (44). This can be done in several ways. If we follow the same method as in (Hallén 1953, p. 431) for the linearized expression, we meet only two differences. They are, 1) the poles of the transformed integral now form an infinite divergent series and 2) the infinite integral, before compensation, diverges. Both difficulties can easily be overcome, and as this way of proceeding leads mainly to Bessel functions J_o , Y_o , of which very dense tables are available, we use this method. Otherwise, it is easy enough to avoid the series, but then the expression will retain functions I_o and K_o which are not so densely tabulated.

I find (cf. Hallén 1953, p. 431) from (44):

$$\varphi_2{}^2(\beta) = \frac{1}{-j\,\pi\, H_o^{(2)}\,(a\beta)} \exp\left\{ \frac{4}{\pi^2} \int_0^1 \frac{\log \frac{1+\sqrt{1-u^2}}{\sqrt{1-u^2}}}{u\, H_o^{(1)}\,(a\beta u)\, H_o^{(2)}\,(a\beta u)}\, du \right. +$$

$$+ \frac{4}{\pi^2} \int_1^\infty \frac{\log \frac{u}{\sqrt{u^2-1}}}{u\, H_o^{(1)}\,(a\beta u)\, H_o^{(2)}\,(a\beta u)}\, du +$$

$$+ \frac{4j}{\pi^2} \int_1^\infty \arcsin\frac{1}{u} \left[\frac{1}{u\, H_o^{(1)}\,(a\beta u)\, H_o^{(2)}\,(a\beta u)} - \frac{\pi a\beta}{2} \right] du +$$

$$+ 2j\,\frac{a\beta}{\pi} \sum_1^\infty \left[\frac{1}{m} - \frac{\pi}{a\beta} \operatorname{arctg} \frac{a\beta}{\sqrt{\xi_m{}^2 - a^2\beta^2}} \right] +$$

$$\left. + 2j\,\frac{a\beta}{\pi} \left[1 - \gamma - \frac{\pi}{2} + \log\frac{2\pi}{a\beta} \right] \right\} \qquad (72)$$

Except for the compensating last bracket term in the last infinite integral and the last terms outside the integrals, which all contain the factor $a\beta$, one recognizes in this expression the corresponding linearized one (Hallén 1953, eqn. 35.58c) if as usual the Hankel functions are substituted by their first logarithmical part with suppressing of powers of $a\beta$. The integrals in (72) are easily converted into others with finite integrands and finite limits (cf. Hallén 1953, eqn. 35.58d). The exact value of the end admittance of the tube-shaped antenna (50) as it follows from (72) is shown in Figure 1, together with the corresponding linearized (68, 69) and the series value as derived from (71). Figure 2 gives a part of the exact curve in another scale.

For the numerical evaluation of the end impedance of the inner current (63) we proceed in the same way, although the formula now differs because of the imaginary value of a in φ_1 in (63). For summation of the series we need an asymptotic formula for $\varphi_1\left(\frac{j}{a}\sqrt{\xi_m^2 - a^2\beta^2}\right)$ for higher values of m. This asymptotic value is found to be:

$$\varphi_1\left(\frac{j}{a}\sqrt{\xi_m^2 - a^2\beta^2}\right) = \frac{1}{\sqrt{\xi_m}}\left(\cos\frac{k_{a\beta}}{\xi_m} - j\sin\frac{k_{a\beta}}{\xi_m}\right) \tag{73}$$

where

$$k_{a\beta} = \frac{a\beta}{\pi}\left\{\operatorname{arctg}\frac{J_0(a\beta)}{-Y_0(a\beta)} - \frac{2}{\pi}\int_0^1 \frac{u\,du}{\left(1+\sqrt{1-u^2}\right) H_0^{(1)}(a\beta u)\, H_0^{(2)}(a\beta u)}\right\}$$

ξ_m is defined by $J_0(\xi_m) = 0$.

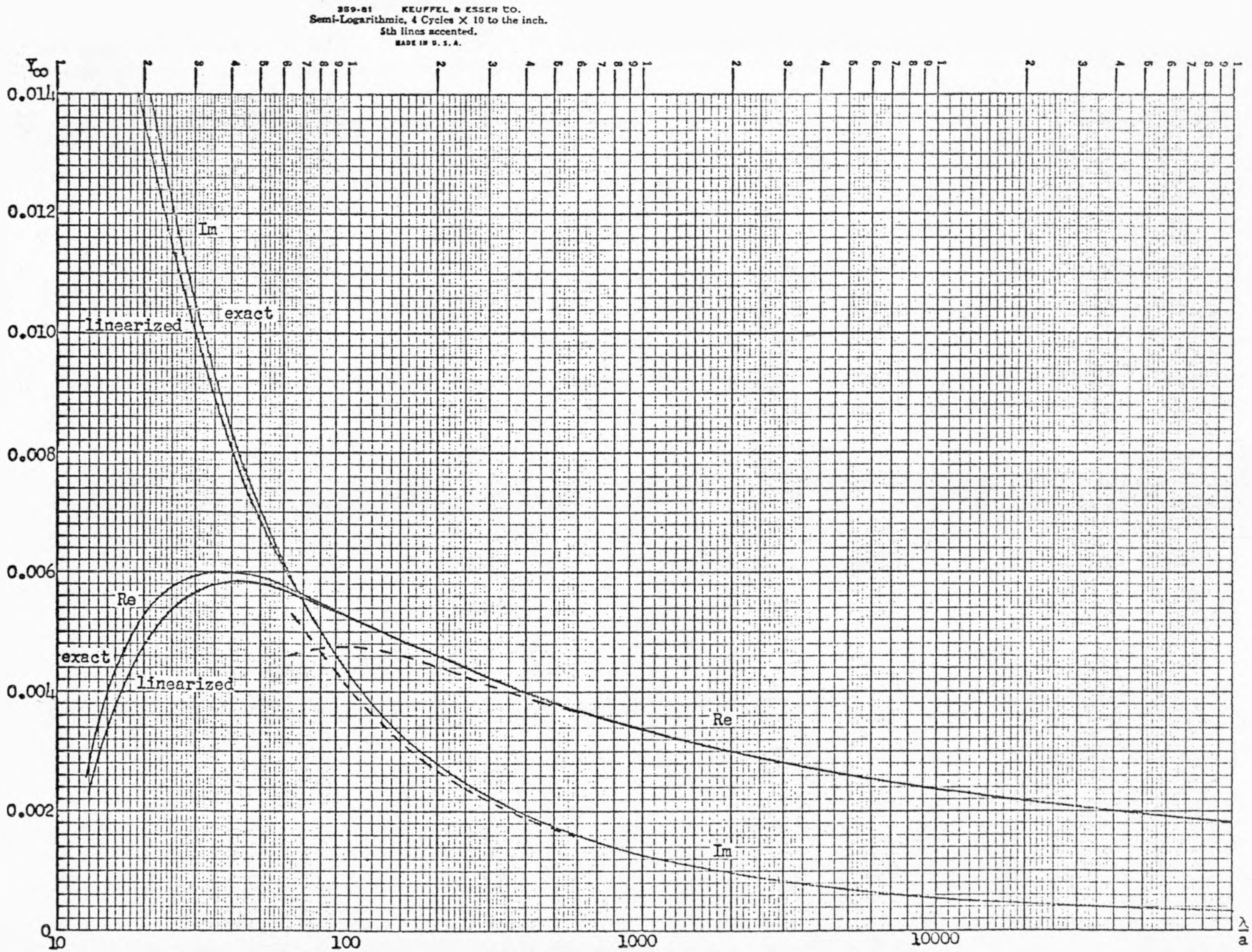

Figure 1
Real and imaginary part of end admittance of tube-shaped antenna, exact value eqn. (67, 72), linearized value (68, 69), and approximate series (68, 71) dashed curve.

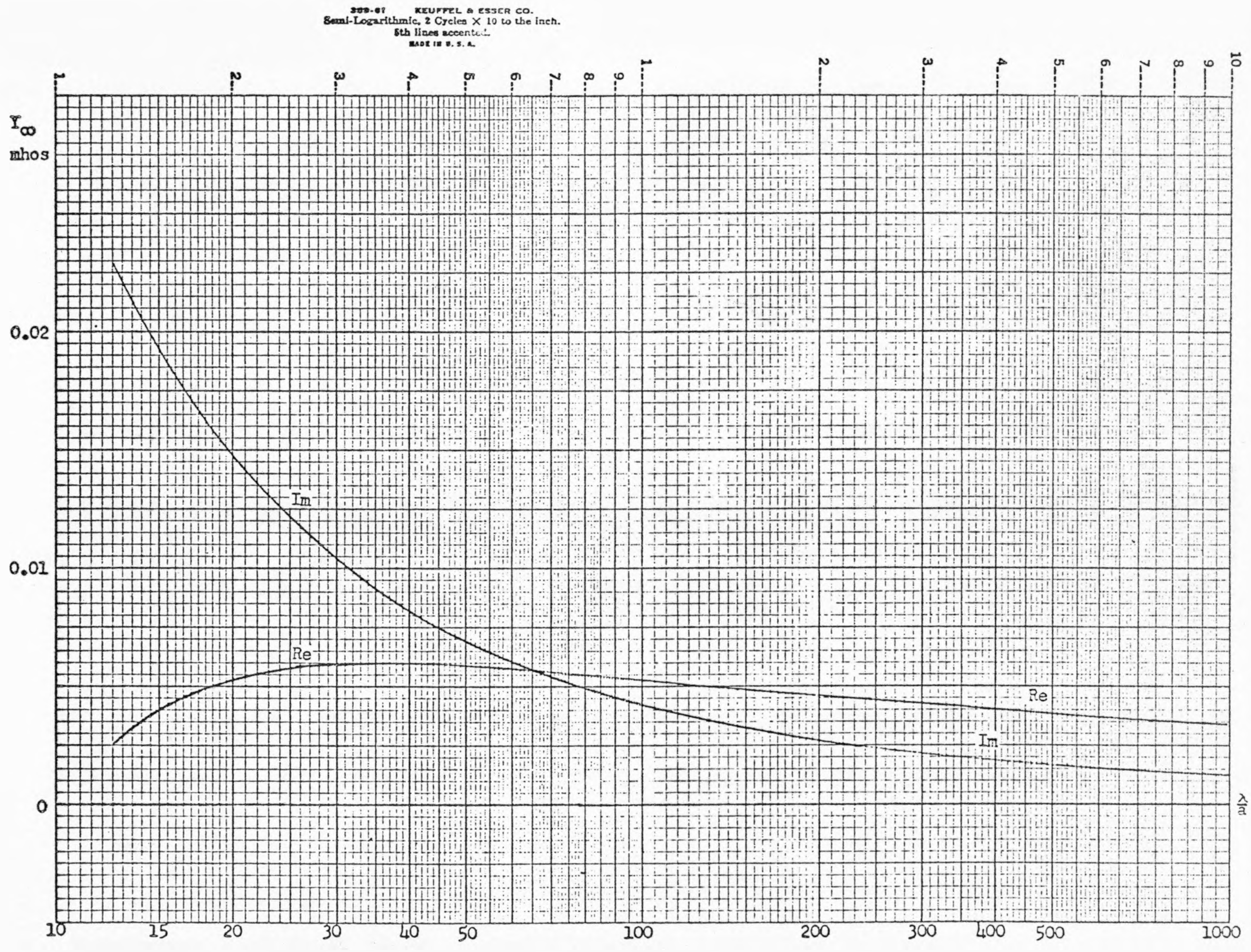

Figure 2
Real and imaginary part of end admittance of tube-shaped antenna, exact value (67, 72).

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